Variational Formulation

inner product : $(u,v) = \int_\Omega u v^* \, d\Omega$

Hilbert Space : $u, v \in \mathcal{H}$

1) $(u,v) = (v,u)^*$
2) $(a_1 u_1 + a_2 u_2, v) = a_1 (u_1, v) + a_2 (u_2, v)$
3) $(u,u) \geq 0$
4) if $(u,u) = 0$ then $u = 0$

Note: $(au, v) = a(u,v)$

$(u, av) = a^*(v,u)^* = a^*(u,v)$

$\therefore \; (au, au) = a(u, au) = aa^*(u,u) = |a|(u,u) > 0$

let L denote an operator over a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$

eg: $Lu = f$ on $\mathcal{D}$

if $L = -\nabla^2$, $f = \frac{\rho}{\epsilon}$ $\Rightarrow$ $-\nabla^2 u = \frac{\rho}{\epsilon}$

on the boundary $\partial\mathcal{D}$ we may have

$u(s) = g(s)$ $\quad s \in \partial\mathcal{D}$ (Dirichlet)

$\left.\frac{\partial u}{\partial n}\right|_s = p(s)$ $\quad s \in \partial\mathcal{D}$ (Neumann)

$\left.\frac{\partial u}{\partial n}\right|_s + \sigma(s) u(s) = q(s)$ $\quad s \in \partial\mathcal{D}$ (mixed or Robbin) we seek solutions to the operator equation from the class of functions which are in the domain of L and satisfy the boundary conditions.

Two other properties are required of the operator:

self-adjoint: $(Lu, v) = (u, Lv)$

positive definite: $(Lu, u) > 0$ if $u \neq 0$

Now consider the functional:

$$F = (Lu, u) - 2(u, f)$$ (Real Hilbert Space)

which maps a fcn u into a number $\in \mathbb{R}$

assume u_0 is the solution of $Lu_0 = f$

then $F = (Lu, u) - 2(u, Lu_0)$

$= (Lu, u) - 2(u, Lu_0) + (Lu_0, u_0) - (Lu_0, u_0)$

$= (Lu, u) - 2(Lu_0, u) + (Lu_0, u_0) - (Lu_0, u_0)$

(by self-adjointness or symmetry) $= (L(u-u_0), u) - (Lu_0, u) + (Lu_0, u_0) - (Lu_0, u_0)$

$= (L(u-u_0), u) - (Lu, u_0) + (Lu_0, u_0) - (Lu_0, u_0)$

$= (L(u-u_0), (u-u_0)) - (Lu_0, u_0)$

Since L is positive definite $(Lu_0, u_0) > 0$
$(L(u-u_0), (u-u_0)) > 0$

$\therefore$ F is minimum at $u = u_0$

$\therefore$ we have the following "minimal functional theorem":

if a linear operator L is self-adjoint and positive definite under the stated boundary conditions then the functional

$$F = (Lu,u) - 2(u,f)$$

assumes its minimum value at $u_0 \in \mathcal{D}_L$ (domain of L) iff u_0 is a solution of

$$Lu_0 = f$$

this minimal value is

$$F_{min} = -(Lu_0, u_0)$$

Note also that if u_0 is a solution, then since L is positive definite it is the unique solution:

say: $Lu_1 = f$ $Lu_2 = f$ are two solutions

since L is linear: $Lu_1 - Lu_2 = L(u_1 - u_2) = Lw = 0$

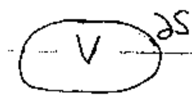
$\therefore \langle Lw, w\rangle = 0$

but since L is positive definite this can only happen if $w \equiv 0 = u_1 - u_2$

$\therefore$ the solution is unique!

example : Poisson's equation

$$L = -\nabla^2 \quad F = \frac{\rho}{\epsilon} \Rightarrow \boxed{-\nabla^2 u = \frac{\rho}{\epsilon}} \quad \text{on } V$$



First we must check for self-adjointness and positive definite properties.

$u, v \in \mathfrak{D}_L$

$$(Lu, v) = -\int_V v \nabla^2 u \, dV$$

using Green's Identity:

$$\int_S v \frac{\partial u}{\partial n} \, ds = \int_V \nabla u \cdot \nabla v \, dV + \int_V v \nabla^2 u \, dV$$

$$\therefore \quad (Lu, v) = \int_V \nabla u \cdot \nabla v \, dV - \int_S v \frac{\partial u}{\partial n} \, ds$$

similarly :

$$(u, Lv) = \int_V \nabla u \cdot \nabla v \, dV - \int_S u \frac{\partial v}{\partial n} \, ds$$

under homogeneous Dirichlet or Neumann boundary conditions the above surface integrals vanish and we have $(u, Lv) = (Lu, v)$ i.e. self-adjoint.

with "mixed - boundary" conditions the two surface integrals become

$$\left.\frac{\partial u}{\partial n}\right|_S + \sigma(s)\,u(s) = 0 \qquad \left.\frac{\partial v}{\partial n}\right|_S + \sigma(s)\,v(s) = 0$$

$$\left.\frac{\partial u}{\partial n}\right|_S = -\sigma(s)\,u(s) \qquad \left.\frac{\partial v}{\partial n}\right|_S = -\sigma(s)\,v(s)$$

$$\therefore -\int_S v\,\frac{\partial u}{\partial n}\,ds = \int_S \sigma(s)\,u(s)\,v(s)\,ds$$

$$= -\int_S u\,\frac{\partial v}{\partial n}\,ds$$

$\therefore$ again we get self-adjointness.

For positive definiteness, let $v = u$

$$\therefore \quad (Lu, u) = \int_V (\nabla u)^2\,dV - \int_S u\,\frac{\partial u}{\partial n}\,ds$$

which is > 0 for $u(s) = 0$ or $\left.\frac{\partial u}{\partial n}\right|_S = 0$

if we have mixed boundary conditions then $\sigma(s) > 0$ is required for positive definite property to hold.

$$F = (-\nabla^2 u, u) - 2\left(u, \frac{\rho}{\epsilon}\right)$$

$$= \int_V \nabla u \cdot \nabla u\,dV - \int_S u\,\frac{\partial u}{\partial n}\,ds - 2\int_V \frac{u\rho}{\epsilon}\,dV$$

which for homogeneous Dirichlet or Neumann boundary conditions becomes:

$$\boxed{F = \int_V |\nabla u|^2 \, dV - 2 \int_V \frac{u\rho}{\epsilon} \, dV} \qquad (1)$$

For the homogeneous mixed boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_S = -\sigma(s)u(s)$$

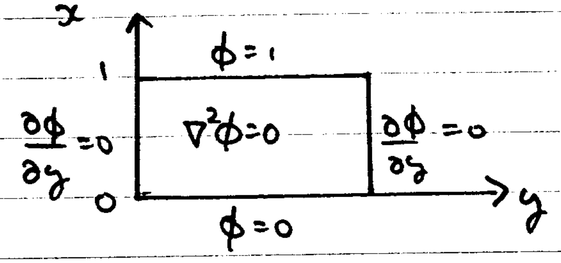
$$\therefore \quad \boxed{F = \int_V |\nabla u|^2 \, dV + \int_S \sigma(s) u^2 \, ds - 2\int_V \frac{u\rho}{\epsilon} \, dV} \qquad (2)$$

Since $\sigma(s) > 0$ $\quad \int_S \sigma(s) u^2 \, ds > 0$

and therefore minimizing (2) is the same as minimizing (1).

Variational Approach to a Solution
The Rayleigh – Ritz Method

Example:



because of the B.C.'s this problem is independent of y i.e. it is a 1-D problem:

$$\frac{d^2\phi}{dx^2} = 0 \qquad \phi(1) = 1 \quad \phi(0) = 0$$

since we are solving Laplace's equation the appropriate Functional to minimize is:

$$\boxed{\mathcal{F}(\phi) = \int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx}$$

this represents the electrostatic energy in the region.

we choose a "trial" function to minimize the Functional with. The trial function must satisfy Dirichlet boundary conditions:

$$\phi(x) = ax^2 + bx + c \qquad \text{(trial function)}$$

B.C.'s :

$$\left.\begin{aligned} \phi(1) &= a + b + c \\ \phi(0) &= c = 0 \end{aligned}\right\} \quad \begin{aligned} c &= 0 \\ b &= 1 - a \end{aligned}$$

$$\therefore \quad \phi(x) = a x^2 + (1-a)x$$
$$= a(x^2 - x) + x$$

where a is the "variational parameter" we will use to minimze the Functional:

$$\mathcal{F}(a) = \int_0^1 \{ a(2x-1)+1 \}^2 \, dx$$

set:

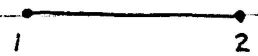
$$0 = \frac{\partial \mathcal{F}(a)}{\partial a} = 2\int_0^1 (2x-1)\left[a(2x-1)+1\right] dx$$
$$= 2\int_0^1 a(2x-1)^2 + (2x-1) \, dx$$
$$= 2a \int_0^1 (2x-1)^2 dx + 2\int_0^1 (2x-1)\, dx$$
$$= 2a X + 0$$
$$\Rightarrow a = 0$$

$\therefore \quad \phi(x) = x$ which just happens to be the exact answer.

This method of choosing a trial function with parameters and then choosing the parameters so as to minimize the Functional is known as the Rayleigh - Ritz Method.

One - Dimensional Linear Interpolation Fcns.

Consider the one-dimensional element with labled "local" node numbers:



the "global" coordinates ar x_1 and x_2

we approximate a fcn $F(x)$ over this element by the linear approximation:

$$F(x) = c_1 + c_2 x \qquad (1)$$

where c_1 and c_2 are constants to be determined. If we know the value of $F(x)$ at x_1 and x_2 then:

$$F(x_1) \equiv F_1 = c_1 + c_2 x_1$$
$$F(x_2) \equiv F_2 = c_1 + c_2 x_2$$

or
$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad (2)$$

$$\left\{ \begin{aligned} c_1 &= \frac{1}{Ae} \left[F_1 x_2 - F_2 x_1 \right] \\ c_2 &= \frac{1}{Ae} \left[F_2 - F_1 \right] \\ Ae &= (x_2 - x_1) \end{aligned} \right. \qquad (3)$$

$$\therefore F(x) = F_1 \left[\frac{x_2 - x}{Ae} \right] + F_2 \left[\frac{x - x_1}{Ae} \right] \qquad (4)$$

$$F(x) = F_1\,\psi_1(x) + F_2\,\psi_2(x) = \sum_{i=1}^{2} F_i\,\psi_i(x) = [F_i]^T[\psi_i] \quad (5)$$
$$= [\psi_i]^T[F_i]$$

$[\alpha_i]$ — represents a column vector

$$\psi_1 = \frac{x_2 - x}{x_2 - x_1} \qquad \psi_2 = \frac{x - x_1}{x_2 - x_1} \qquad \text{(shape Fcns)} \quad (6)$$

<u>properties</u> of linear shape Fcns:

$$\psi_i(x_j) = \delta_{ij} \qquad i,j = 1,2$$

(i.e. equal to 1 at end-point)

$$\sum_{i=1}^{2} \psi_i(x) = 1$$ (sum produces a constant of 1)

$\sum_{i=1}^{2} F_i\,\psi_i(x)$ represents a line through (x_1, F_1) and (x_2, F_2).

example

For the sample problem

$$\begin{cases} -\dfrac{d^2 f}{dx^2} = 1 + 4x^2 & 0 \le x \le 1 \\ f(0) = f(1) = 0 \end{cases}$$

we minimize the quadratic functional over elements:

$$I(f) = \int_{x_1}^{x_2} \left(\frac{df}{dx}\right)^2 dx - 2\int_{x_1}^{x_2} f\,(1+4x^2)\,dx$$

$$f^{(e)} = [f_i]^T[\psi_i] = [\psi_i]^T[f_i]$$

$$\frac{df^{(e)}}{dx} = [f_i]^T[\psi_i'] = [\psi_i']^T[f_i]$$

$$[\psi_i'] = \begin{bmatrix} \dfrac{1}{x_1 - x_2} \\ \dfrac{1}{x_2 - x_1} \end{bmatrix}$$

$$\therefore\ I(f) = [f_i]^T \int_{x_1}^{x_2} [\psi_i'][\psi_i']^T dx\,[f_i] - 2\int_{x_1}^{x_2} [\psi_i]^T(1+4x^2)\,dx\,[f_i]$$

$$= [f_i]^T\, S\, [f_i] - 2\,[f_i]^T\,\underline{b}$$

$$S = \int_{x_1}^{x_2} [\psi_i'][\psi_i']^T dx = \int_{x_1}^{x_2} \begin{bmatrix} \psi_1'\psi_1' & \psi_1'\psi_2' \\ \psi_2'\psi_1' & \psi_2'\psi_2' \end{bmatrix} dx$$

$$\underline{b} = \int_{x_1}^{x_2} [\psi_i]\,(1+4x^2)\,dx = \int_{x_1}^{x_2} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} (1+4x^2)\,dx$$

minimizing the integral : denoting $[f_i] = \underline{f}$

$$\frac{\partial I(f)}{\partial \underline{f}} = \frac{\partial}{\partial \underline{f}}\left[\underline{f}^T S \underline{f} - 2\underline{f}^T \underline{b} \right]$$

$$= S\underline{f} + \left[\frac{\partial (S\underline{f})}{\partial \underline{f}}\right]^T \underline{f} - 2\frac{\partial \underline{f}^T}{\partial \underline{f}}\underline{b}$$

$$= S\underline{f} + \frac{\partial \underline{f}^T}{\partial \underline{f}} S^T \underline{f} - 2b$$

$$= 2S\underline{f} - 2\underline{b} = 0 \qquad \text{(since } S \text{ is symmetric)}$$

$\therefore$ we have the matrix equation

$$S\underline{f} = \underline{b} \qquad \text{for every element}$$

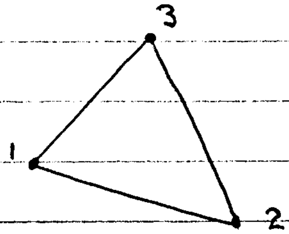
$$S = \int_{x_1}^{x_2} \begin{bmatrix} \frac{1}{(x_1-x_2)^2} & \frac{-1}{(x_2-x_1)} \\ \frac{-1}{(x_2-x_1)} & \frac{1}{(x_1-x_2)^2} \end{bmatrix} dx = \frac{1}{x_2-x_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{b} = \int_{x_1}^{x_2} \begin{bmatrix} \frac{x_2-x}{x_2-x_1} \\ \frac{x-x_1}{x_2-x_1} \end{bmatrix} (1+4x^2)\, dx = \frac{1}{(x_2-x_1)}\begin{bmatrix} -\frac{1}{2}x^2 + \frac{4}{3}x_2 x^3 - x^4 \\ \frac{1}{2}x^2 - \frac{4}{3}x_1 x^3 + x^4 \end{bmatrix}_{x_1}^{x_2}$$

$$= \begin{bmatrix} -\frac{1}{2}(x_2-x_1) + \frac{4}{3}x_2(x_2-x_1)^2 - (x_2-x_1)^3 \\ \frac{1}{2}(x_2-x_1) - \frac{4}{3}x_1(x_2-x_1)^2 + (x_2-x_1)^3 \end{bmatrix}$$

etc...

Two - Dimensional Interpolation Functions



Consider the triangular element where the "local" node numbers are labled <u>counterclockwize</u> with "global" coordinates:

$$(x_1, y_1), (x_2, y_2), (x_3, y_3)$$

now say we want to approximate a fcn $f(x,y)$ over the domain of the triangle by the linear fcn:

$$f(x,y) = c_1 + c_2 x + c_3 y \qquad (1)$$

where c_1, c_2, and c_3 are constants to be determined.

now say we knew the value of $f(x,y)$ at the three vertices of the triangle, that is:

$$f_1 \equiv f(x_1, y_1) = c_1 + c_2 x_1 + c_3 y_1$$
$$f_2 \equiv f(x_2, y_2) = c_1 + c_2 x_2 + c_3 y_2$$
$$f_3 \equiv f(x_3, y_3) = c_1 + c_2 x_3 + c_3 y_3$$

or

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad (2)$$

by Cramer's Rule we can solve For the constants c_1, c_2, c_3 :

$$c_1 = \frac{\begin{vmatrix} f_1 & x_1 & y_1 \\ f_2 & x_2 & y_2 \\ f_3 & x_3 & y_3 \end{vmatrix}}{2A_e} \qquad c_2 = \frac{\begin{vmatrix} 1 & f_1 & y_1 \\ 1 & f_2 & y_2 \\ 1 & f_3 & y_3 \end{vmatrix}}{2A_e}$$

$$c_3 = \frac{\begin{vmatrix} 1 & x_1 & f_1 \\ 1 & x_2 & f_2 \\ 1 & x_3 & f_3 \end{vmatrix}}{2A_e} \qquad 2A_e = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\begin{cases} c_1 = \dfrac{1}{2A_e}\left[f_1(x_2y_3 - x_3y_2) + f_2(x_3y_1 - x_1y_3) + f_3(x_1y_2 - x_2y_1)\right] \\ \\ c_2 = \dfrac{1}{2A_e}\left[f_1(y_2 - y_3) + f_2(y_3 - y_1) + f_3(y_1 - y_2)\right] \\ \\ c_2 = \dfrac{1}{2A_e}\left[f_1(x_3 - x_2) + f_2(x_1 - x_3) + f_3(x_2 - x_1)\right] \end{cases} \quad (3)$$

permutation

$$2A_e = (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1) \quad (4)$$

substituting (3) and (4) into (1) :

$$F(x,y) = f_1\psi_1(x,y) + f_2\psi_2(x,y) + f_3\psi_3(x,y)$$
$$= \sum_{i=1}^{3} f_i\psi_i = [f_i]^T[\psi_i] = [\psi_i]^T[f_i] \quad (5)$$

$[\alpha_i]$ - represents a column vector the ψ_i are called "basis", "shape" or "interpolation" Fcns.

$$\psi_i = \frac{1}{2A_e}\left(\alpha_i + \beta_i x + \gamma_i y\right) \qquad i = 1,2,3 \qquad (6)$$

$$\begin{cases} \alpha_i = (x_j y_k - x_k y_j) \\ \beta_i = (y_j - y_k) \\ \gamma_i = (x_k - x_j) \end{cases} \qquad i \to j \to k \to i \quad \text{"natural" permutation cycle} \qquad (7)$$

eg: $\alpha_2 \Rightarrow \quad i=2 \quad j=3 \quad k=1$ in (7)

<u>properties</u> of linear shape Fcns:

$$\psi_i(x_j, y_j) = \delta_{ij} \qquad i,j = 1,2,3 \qquad (8)$$

(i.e. equal to 1 at triangle nodes)

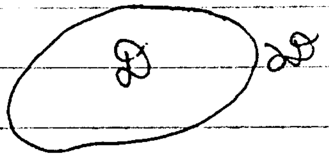
$$\sum_{i=1}^{3} \psi_i = 1 \qquad (9)$$

(i.e. adding them with equal weight produces a constant 1 over the triangle).

$\sum_{i=1}^{3} F_i \psi_i$ represents a plane passing through (x_1, y_1, F_1), (x_2, y_2, F_2) and (x_3, y_3, F_3)

Laplace's Equation : $\nabla^2 \phi = 0$ on $\mathcal{D}$ ①

$\phi(s) = g(s)$ $s \in \partial\mathcal{D}$ ②



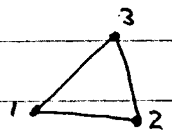
(Dirichlet problem)

solution of ① is equivalent to minimization of quadratic Functional

$$I(\phi) = \frac{1}{2}\int_{\mathcal{D}} |\nabla\phi|^2 \, dV \quad ③$$

we now approximate ϕ on each triangular element as

$$\phi^{(e)}(x,y) = \underline{\phi}^T \underline{\alpha} = \underline{\alpha}^T \underline{\phi} \quad ④$$



$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad \underline{\alpha} = \begin{bmatrix} \alpha_1(x,y) \\ \alpha_2(x,y) \\ \alpha_3(x,y) \end{bmatrix}$$

$\underline{\phi}$ is column vector of vertex values

$\underline{\alpha}$ is column vector of "shape Fcns"

aside: vector differentiation notation

$\underline{a}^T = [a_1 \; a_2 \ldots a_n]$

$$\left.\begin{aligned} \frac{\partial \underline{a}^T}{\partial a_1} &= [1 \; 0 \ldots 0] \\ \frac{\partial \underline{a}^T}{\partial a_2} &= [0 \; 1 \ldots 0] \\ &\vdots \\ \frac{\partial \underline{a}^T}{\partial a_n} &= [0 \; 0 \ldots 1] \end{aligned}\right\} \quad \frac{\partial \underline{a}^T}{\partial \underline{a}} = I$$

if: $y = \underline{b}^T \underline{c}$

then: $\dfrac{\partial y}{\partial \underline{a}} = \dfrac{\partial(\underline{b}^T \underline{c})}{\partial \underline{a}}$

$$\boxed{\frac{\partial y}{\partial \underline{a}} = \frac{\partial \underline{b}^T}{\partial \underline{a}}\underline{c} + \frac{\partial \underline{c}^T}{\partial \underline{a}}\underline{b}}$$

prove this to yourself.

substituting ④ into ③:

$$I(\phi) = \frac{1}{2}\int_{\mathcal{D}} \phi^T(\nabla\underline{\alpha}\cdot\nabla\underline{\alpha}^T)\underline{\phi}\, dV$$

$$I(\phi) = \frac{1}{2}\underline{\phi}^T \int_{\mathcal{D}} (\nabla\underline{\alpha}\cdot\nabla\underline{\alpha}^T)\, dV\, \underline{\phi}$$

$$I(\phi) = \frac{1}{2}\underline{\phi}^T S\, \underline{\phi} \qquad S = \int (\nabla\underline{\alpha}\cdot\nabla\underline{\alpha}^T)\, dV \quad (5)$$

minimizing $I(\phi)$:

$$0 = \frac{\partial I}{\partial \underline{\phi}} = \frac{1}{2}\left[\frac{\partial \underline{\phi}^T}{\partial \underline{\phi}}(S\underline{\phi}) + \frac{\partial (S\phi)^T}{\partial \underline{\phi}}\underline{\phi}\right]$$

$$= \frac{1}{2}\left[S\underline{\phi} + \frac{\partial}{\partial \underline{\phi}}(\underline{\phi}^T S^T)\,\underline{\phi}\right]$$

$$= \frac{1}{2}\left[S\underline{\phi} + \frac{\partial \underline{\phi}^T}{\partial \underline{\phi}} S^T\underline{\phi}\right] = \frac{1}{2}\left[S\underline{\phi} + S^T\underline{\phi}\right]$$

now since S is symmetric

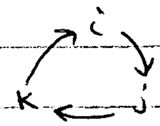
$$\frac{\partial I}{\partial \underline{\phi}} = S\underline{\phi} = 0 \qquad (6)$$

⑥ defines the contribution of each triangle to the final matrix equation.

For First order "shape" Fcns $\alpha_i(x,y)$

$$\alpha_i = \frac{1}{2A_e}\left(a_i + b_i x + c_i y\right) \qquad i = 1,2,3$$

$$\begin{cases} a_i = (x_j y_k - x_k y_j) \\ b_i = (y_j - y_k) \\ c_i = (x_k - x_j) \end{cases}$$



$$\nabla \alpha_i = \frac{\partial \alpha_i}{\partial x}\hat{x} + \frac{\partial \alpha_i}{\partial y}\hat{y}$$

$$= \frac{1}{2A_e}\left[b_i \hat{x} + c_i y\right]$$

$$\nabla \underline{\alpha} \cdot \nabla \underline{\alpha}^T = \frac{1}{4A_e^2}\begin{bmatrix} \nabla\alpha_1\cdot\nabla\alpha_1 & \nabla\alpha_1\cdot\nabla\alpha_2 & \nabla\alpha_1\cdot\nabla\alpha_3 \\ \nabla\alpha_2\cdot\nabla\alpha_1 & \nabla\alpha_2\cdot\nabla\alpha_2 & \nabla\alpha_2\cdot\nabla\alpha_3 \\ \nabla\alpha_3\cdot\nabla\alpha_1 & \nabla\alpha_3\cdot\nabla\alpha_2 & \nabla\alpha_3\cdot\nabla\alpha_3 \end{bmatrix}$$

$$= \frac{1}{4A_e^2}\begin{bmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_1 & b_3 b_3 + c_3 c_3 \end{bmatrix}$$

$$2A_e = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)$$

$$\therefore \quad S = \int_\Delta \nabla\underline{\alpha}\cdot\nabla\underline{\alpha}^T \, dV = \nabla\underline{\alpha}\cdot\nabla\underline{\alpha}^T \int_\Delta dV = A_e \nabla\underline{\alpha}\cdot\nabla\underline{\alpha}^T$$

$$\therefore \quad \boxed{S = \frac{1}{4A_e}\begin{bmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3 b_3 + c_3 c_3 \end{bmatrix}}$$

Eigenvalue Problem:

$$\nabla^2 \phi + K_c^2 \phi = 0 \qquad \text{on } \mathscr{D}$$

$$\phi(s) = 0 \qquad s \text{ on } \partial\mathscr{D} \quad \text{(TM modes)}$$

$$\left.\frac{\partial \phi}{\partial n}\right|_s = 0 \qquad s \text{ on } \partial\mathscr{D} \quad \text{(TE modes)}$$

$$K_c = \frac{2\pi}{\lambda_c} \qquad \lambda_c = \text{cut-off wavelength}$$

Functional is:

$$I(\phi) = \frac{1}{2} \int_{\mathscr{D}} \left[(\nabla\phi)^2 - K_c^2 \phi^2 \right] dx\,dy$$

again letting $\phi^{(e)}(x,y) = \underline{\phi}^T \underline{\alpha} = \underline{\alpha}^T \underline{\phi}$ (over a triangle)

$$I(\phi) = \frac{1}{2} \int_{\Delta} \left[\underline{\phi}^T (\nabla\underline{\alpha} \cdot \nabla\underline{\alpha}^T) \underline{\phi} - K_c^2 \underline{\phi}^T \underline{\alpha}\,\underline{\alpha}^T \underline{\phi} \right] dx\,dy$$

$$= \underline{\phi}^T \frac{1}{2} \int_{\Delta} \nabla\underline{\alpha} \cdot \nabla\underline{\alpha}^T \, dx\,dy - \frac{K_c^2}{2} \underline{\phi}^T \int_{\Delta} \underline{\alpha}\,\underline{\alpha}^T \, dx\,dy \, \underline{\phi}$$

$$= \frac{1}{2} \left[\underline{\phi}^T S \underline{\phi} - K_c^2 \underline{\phi}^T T \underline{\phi} \right]$$

$$S = \int_{\Delta} \nabla\underline{\alpha} \cdot \nabla\underline{\alpha}^T \, dx\,dy \qquad T = \int_{\Delta} \underline{\alpha}\,\underline{\alpha}^T \, dx\,dy$$

S is the same as for the Laplace equation.

minimizing $I(\phi)$:

$$\frac{\partial I}{\partial \underline{\phi}} = 0 = S\underline{\phi} - \frac{\partial}{\partial \underline{\phi}}\left(\frac{K_c^2}{2}\underline{\phi}^T T \underline{\phi}\right)$$

$$= S\underline{\phi} - \frac{K_c^2}{2}\left[\frac{\partial \underline{\phi}^T}{\partial \underline{\phi}}(T\underline{\phi}) + \frac{\partial}{\partial \underline{\phi}}(T\underline{\phi})^T \underline{\phi}\right]$$

$$= S\underline{\phi} - \frac{K_c^2}{2}\left[T\underline{\phi} + \frac{\partial}{\partial \underline{\phi}}(\underline{\phi}^T T^T)\underline{\phi}\right]$$

$$= S\underline{\phi} - \frac{K_c^2}{2}\left[T\underline{\phi} + T^T\underline{\phi}\right]$$

$$0 = S\underline{\phi} - K_c^2 T\underline{\phi} \qquad (T \text{ is symmetric})$$

$$\boxed{S\underline{\phi} = K_c^2 T\underline{\phi}} \qquad \left(\text{Matrix eigenvalue equation}\right)$$

we now must evaluate the matrix T:

$$T = \int_{\Delta} \underline{\alpha} \cdot \underline{\alpha}^T \, dV = \frac{A_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$